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XIII.

ON THE LIMITS OF ACCURACY IN MEASUREMENTS WITH THE TELESCOPE AND THE MICROSCOPE.

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Presented Oct. 9, 1878.

IT is often desirable in astronomical observations to assign to a given result the degree of precision which the observations will justify. Usually the limit of precision is defined either by the probable error of a single observation, or of the mean of a given number of observations.

Let

$x =$ any given numerical value.
$n =$ the number of values of x.
$v =$ the difference between each value of x and the arithmetical mean of all the values.
$[v] =$ the sum of the separate residuals, without regard to sign.
$r =$ the probable error of a single value.
$r_0 =$ the probable error of the arithmetical mean.

We shall then have, —

$$r = .6745\sqrt{\frac{[v\,v]}{m-1}}$$
$$r_0 = .6745\sqrt{\frac{[v\,v]}{m\,(m-1)}} \qquad (a)$$

Or, according to Peters, —

$$r = .8453\ \frac{[v]}{\sqrt{m\,(m-1)}}$$
$$r_0 = .8453\ \frac{[v]}{m\sqrt{(m-1)}} \qquad (b)$$

As an illustration, we assume the following values of x, *without defining their signification*:—

x	v	vv
″	″	″
61.70	+ .05	+ .02
61.50	+ .25	+ .06
60.90	+ .85	+ .72
61.70	+ .05	+ .02
61.30	+ .45	+ .20
61.20	+ .55	+ .30
60.80	+ .95	+ .90
61.90	— .15	+ .02
61.60	+ .15	+ .03
61.50	+ .25	+ .06
62.80	— 1.05	+ 1.10
62.70	— .95	+ .90
60.80	+ .95	+ .90
61.30	+ .45	+ .20
63.00	— 1.25	+ 1.56
61.10	+ .65	+ .42
63.90	— 2.15	+ 4.62
Mean, 61.75		

From equations (a) we have,—

$$r = \pm .6745 \sqrt{\frac{12.03}{16}} = \pm .585''$$

$$r_0 = \pm .6745 \sqrt{\frac{12.03}{16 \times 17}} = \pm .142''$$

And from equations (b),—

$$r = \pm \frac{.8453 \times 11.15}{\sqrt{17 \times 16}} = \pm .571''$$

$$r_0 = \pm \frac{.8453 \times 11.15}{17 \sqrt{16}} = \pm .139''$$

If we reject the last value of x, viz. $63''.90$, we have:—

From (a),—

$$r = \pm \overset{''}{.474}$$

$$r_0 = \pm .119$$

From (b),—

$$r = \pm .491$$

$$r_0 = \pm .123$$

Let us now inquire what interpretation can be safely given to these values of r and r_0, and what conclusions can be drawn therefrom concerning the precision of x.

First, it will be seen that the two formulæ do not give precisely the same results, and the relation between the results is changed by rejecting one apparently discordant observation. The difference is, however, insignificant when compared with the actual error of observation. In general, the agreement will be more perfect the greater the number of values of x.

Second, it is obvious that if for x we write, $x \pm$ a constant, the values of v will not be thereby changed; hence the values of r and r_0 *will give no indication whatever with reference to the existence of any constant error involved in the values of* x.

Admitting, then, that there is no constant error in the given series, what degree of precision can be assigned to any single value of x, and to the mean value $61''.75$?

It would hardly seem necessary to call attention to the erroneous assumption that, since the value of r is $\pm .57''$, therefore no single value can be greater than $62''.32$, nor less than $61''.18$; or that since r_0 is $\pm .14''$, therefore the value $61''.75$ is true within this limit. The refutation of the first assumption is made sufficiently easy by an examination of the separate values of x, but it is not quite so easy to show the fallacy of the second.

Notwithstanding the absurdity of attempting to assign to the arithmetical mean the degree of precision indicated by the value of r_0, observers of limited experiences are continually found doing this, and the writer recalls two instances in which professional astronomers have committed themselves to the same fallacy.

In general, it is entirely unsafe to draw conclusions with respect to the degree of precision to be attached to the arithmetical mean from the magnitude of the probable error, until the signification of the values from which it is derived is defined.

If the values of x are found by successive readings of the four microscopes of a meridian circle for the same position of the telescope, the separate values are *simple functions* of the quantity required, and involve only the accidental errors of the observer, either in making the bisections of the divisions of the circle, or in reading the index of the micrometer screws. In this case the probable error of the mean is a tolerably accurate indication of the degree of precision which may be attached to it.

But the values of x given, represent the observed index errors of the meridian circle of Harvard College Observatory, as derived from separate fundamental stars, observed January 5, 1872. They are given on page xxiii., Vol. X., of the Annals of the Observatory. Here x is a *complex function.* It involves not only the error of reading the microscopes, but several other *classes* of errors. They may be enumerated as follows: —

Errors depending on the observer.	(*a*) Error of reading microscopes.
	(*b*) Error of bisection of the star observed, or its equivalent.
Errors depending on the instrument.	(*c*) Errors of graduation of the circle, both accidental and systematic.
	(*d*) Error depending on the micrometer screws of microscopes.
	(*e*) Error due to the flexure of the instrument.
	(*f*) Error due to an imperfect figure of the pivots.
	(*g*) Error resulting from a change in the position of the instrument during the observations.
Errors independent of the observer and of the instrument.	(*h*) Error resulting from an erroneous place of the fundamental star observed.
	(*i*) Error depending on the state of the atmosphere, including the constant of refraction, imperfect thermometers, barometers, &c.

In this case, then, one must place quite a different interpretation upon the probable error of the mean value. In fact, the only safe interpretation that can be given to it, is the one which regards it as a means of comparing observations made by different observers under nearly the same conditions and in the same manner.

This subject may be considered in another way. It is a property of the arithmetical mean that it makes the sum of the squares of the residuals a minimum. The solution of a greater number of equations than the unknown quantities which they contain, by the process of least squares, rests upon the same basis; viz. that such values must be given to the unknown quantities as will, when substituted in the original equations, make the sum of the squares of the residuals a minimum. Theoretically, any unknown quantity may be made equal to a constant plus the sum of all the corrections which make up this quantity. We may always have

$$X = C + Aa + Bb + Cc + Dd, \text{ \&c.}$$

The only limit to the number of terms is the one which requires that the coefficients A, B, C, D, &c. shall be known. The solution of a series of equations of this form will give the most probable values of the constant C, and of the unknown quantities a, b, c, d, &c., provided

x is a simple function; but if x is a complex function the solution will no longer give the *true* values of the separate unknown quantities, though it may yield such values as will give the most probable sum of $A\,a$, $B\,b$, $C\,c$, $D\,d$, &c., with respect to their effect upon x.

Let us take, as an illustration, the ordinary equation for the reduction of transit observations. The fundamental equation may be put under the following variety of forms: —

$$\begin{array}{ll}
(a) & 0 = \Delta\mathrm{T} + [T - \mathrm{R.\,A.}] + A\,a \\
(b) & = \Delta\mathrm{T} + [T - \mathrm{R.\,A.}] + A\,a + B\,b \\
(c) & = \Delta\mathrm{T} + [T - \mathrm{R.\,A.}] + A\,a + B\,b + C\,c \\
(d) & = \Delta\mathrm{T} + [T_0 - \mathrm{R.\,A.}] + \tau\,h + A\,a + B\,b + C\,c \\
(e) & = \Delta\mathrm{T} + [T_0 - \mathrm{R.\,A.}] + \tau\,h + A\,a + B\,b + C\,c + D\,d \\
(f) & = \Delta\mathrm{T} + [T_0 - \mathrm{R.A.}_0] + E\,e + \tau\,h + A\,a + B\,b + C\,c + D\,d \\
(g) & = \Delta\mathrm{T} + [T_0 - \mathrm{R.A.}_0] + E\,e + \tau\,h + Aa + Bb + Cc + Dd + C
\end{array}$$

If the level b and the collimation c are obtained independently of the observations by direct measures, then, neglecting the small terms which follow, for any time, T, and with the given right ascension, R. A., the only unknown quantities in equation (a) are the clock error ΔT and the azimuth term $A\,a$. A solution of a series of equations of this form will give the most probable *individual* values of a and ΔT.

If the level term $B\,b$ is unknown, the general equation takes the form (b). Notwithstanding the fact that the equation is somewhat more complex in its structure, the solution by least squares will give the most probable individual values of a and b, if the stars are selected with reference to a proper distribution of positive and negative values for A and B.

If the collimation term $C\,c$ is unknown, the equation takes the form (c). Here a solution by least squares will *not* give the most probable individual values of a, b, and c, unless the observations are arranged with proper reference both to the magnitude and the sign of A, B, and C. Even when these precautions are observed, the value of c from the solution will rarely agree exactly with the value obtained from reversal or from collimators.

If, for any star, the observed time T is written $T_0 + \tau\,h$, the term $\tau\,h$ being the hourly rate of the clock multiplied by the interval τ between T and T_0, the equation takes the form (d). We now introduce an unknown quantity depending on another instrument, viz. the clock.

We may still further introduce the term $D\,d$, representing the diurnal aberration, giving the form (e), and by substituting R. $A._0 + E\,e$ for R. A. where $E\,e$ represents a term depending on 2 ☽, we get the form (f).

Finally, if we represent by the constant C the personal equation between bright and faint wires, bright and faint stars, &c., we have the form (g).

Of course it is wholly absurd to introduce the terms $D\,d$, $E\,e$, and C as unknown quantities, and these forms are given only to show that one must exercise sound judgment in the formation of the equations in order that the solution by least squares shall give correct results. It is useless to expect that the solution will separate errors which appertain to different instruments. For example, in form (g) it would seem hardly necessary to say that the solution will entirely fail in assigning to the telescope the correct values of a, b, and c; to the clock, the true values of ΔT and $\tau\,h$, to yield the physical constant which enters into the diurnal aberration, and the coefficient which results from the variable motion of the moon; and to refer to the observer the constant which involves the various forms of personal equation. Yet, according to the common acceptation of the theory, this form of the equation is allowable, since all the unknown quantities have known coefficients.

Again, as soon as the equation involves unknown quantities which pertain to different instruments, it becomes so complex in its character that we can no longer assume that even the *sum* of the terms which affect ΔT is the most probable value that can be found, for in so doing we assume that ΔT is a constant, whereas the solution requires it to be a variable.

Let us now inquire how far these views are confirmed by the facts of observation.

In my own case, the probable error of a *single* reading of four microscopes of the meridian circle is $\pm .094''$. If, therefore, as many as 10 observations are obtained, the probable error of the mean will be not far from $\pm .03''$. The probable error of a single difference between myself and my assistant, Mr. Joseph F. MacCormick, is for a single reading of four microscopes $\pm .125''$.

The probable error of a *single* complete observation in declination is, in my own case, about $\pm .36''$, and of the mean of 10 observations is $\pm .11''$. The probable error of a *single* complete observation in right ascension is, for an equational star, $\pm .026^{s.}$ and for the mean of 10 observations $\pm .008^{s.}$.

If, therefore, the probable error can be taken as a measure of the accuracy of the observations, there ought to be no difficulty in obtaining, from a moderate number of observations, the right ascension within $.02^{\text{s.}}$ and the declination within 0″.2. Yet it is doubtful, after continuous observations in all parts of the world for more than a century, if there is a single star in the heavens whose absolute coordinates are known within these limits. In 1866 the illustrious Argelander proposed a list of stars for simultaneous observation by different observers, for the purpose of investigating the systematic differences which he found to exist in all modern catalogues. This scheme was carried out only to a limited extent. But in 1878 the fortunate requirements of a special problem secured data which will go far towards the establishment of the existence of these errors, even with the present methods of refinement in observation, if indeed they do not for the present reveal their cause.

During that year Mr. David Gill, recently appointed Director of the Cape of Good Hope Observatory, solicited the co-operation of astronomers in determining the co-ordinates of 28 stars, which he used in his heliometer observations of the planet Mars for obtaining the solar parallax. The observatories named below made the observations required, which were forwarded to Mr. Gill upon the completion of the reductions. The results are published in Vol. XXXIX., page 99, of the Monthly Notices of the Royal Astronomical Society.

In the following table are given the differences between the least and the greatest results for each star, both in right ascension and in declination.

STAR.	$\Delta\alpha$	$\Delta\delta$	STAR.	$\Delta\alpha$	$\Delta\delta$	STAR.	$\Delta\alpha$	$\Delta\delta$
	s.	″		s.	″		s.	″
1	0.189	1.45	10	0.460	2.84	19	0.242	3.14
2	.169	2.04	11	.287	1.77	20	.333	2.70
3	.377	2.63	12	.077	1.74	21	.220	2.16
4	.098	2.57	13	.350	1.80	22	.283	2.31
5	.224	2.04	14	.263	2.34	23	.260	1.77
6	.166	1.86	15	.270	1.80	24	.203	2.78
7	.193	3.47	16	.183	1.86	25	.219	1.62
8	.190	1.93	17	.225	3.47	26	.207	2.06
9	.300	3.15	18	.287	1.37	27	.298	2.57
						28	.264	2.47

Even after the observations were reduced to a homogeneous system, Mr. Gill finds the following outstanding errors: —

AUTHORITY.	Δ α	Δ δ	AUTHORITY.	Δ α	Δ δ
	s.	″		s.	″
Königsberg,	+.005	—0.71	Leiden,	—.053	—0.19
Melbourne,	+.026	—0.49	Paris,	+.055	+0.01
Pulkowa,	+.005	+0.36	Washington,	—.120	+0.78
Leipzig,	+.049	+0.40	Harvard College,	—.072	+0.09
Greenwich,	+.009	—0.56	Cordoba,	—.032	—0.20
Berlin,	+.044	+0.67	Oxford,	+.076	+0.21

These systematic discordances, especially in right ascension, are so alarmingly large that, unless they can be reconciled, the heliometer observations are comparatively worthless. Mr. Gill, therefore, proposed a second list of 12 stars, one half comparatively bright and the other half faint. The observations of these stars are now completed, but the only series yet at hand, are those of Königsberg and Harvard College. Here the discordance is very large, and varies with the magnitude of the star observed. Professor Pickering, the Director of Harvard College Observatory, early in this investigation, proposed the artificial reduction of the magnitude of the bright stars by holding circular diaphragms of varying diameters in front of the object-glass of the telescope. By alternating between bright and faint images of the same star, on different groups of the transit threads, the personal equation between bright and faint stars can be found. This plan was followed in the investigation at Harvard College Observatory, at Leiden, and probably at some other observatories. At Harvard College Observatory, also, a sensible difference was found between results obtained with bright and faint fields of the telescope, this difference varying with the magnitude of the star.

A similar investigation is now being made in another class of observations, viz. the measurement of the position angle and distance of double stars with the filar-micrometer. The range of systematic discordances between the measures of different observers is of course here far less than will always be found in the determination of position in space, for such observations are entirely relative in their character; but the outstanding errors are still so large as to demand a special investigation. Even with observers of skill and long experience, such as Struve, Hall, Dembowski, Burnham, and Stone, there are residual errors in the measurements of the same components far exceeding the limits indicated by the magnitude of the probable error of any single observer.

Finally, it is even an open question whether any real advance has been made in the absolute precision of observations with the telescope for the last forty years, if we except the tentative investigations of the last five or six years. Argelander's Åbo Catalogue of 1830, and the Pulkowa Catalogue of 1845, are as yet pre-eminent for that kind of accuracy which answers to the crucial test of agreement with future observations. After a lapse of nearly fifty years, Argelander's positions of the thirty-six stars known as the "Maskelyne fundamental stars" are at least as near the truth as the mean of the observations of these stars made during the last ten years.

The great need of instrumental astronomy is a rigid investigation of all the classes of error to which observations are now subject, not simply for any one observer, but for all the principal observers of the world, and *upon a common plan.* If some competent and recognized authority, like the *Astronomischen Gesellschaft,* would arrange a scheme of observations having this object in view, and take measures to secure the co-operation of all the principal observatories in this work, it would seem that the foundation for a real advance might be made in the precision with which observations can be made.

In the investigation which follows I have endeavored to ascertain the limits of accuracy in measurements with the microscope by a process similar to that by which observers with the telescope are now seeking to reach the ultimate limit of precision. The remarks already made with regard to the degree of reliability to be attached to conclusions drawn from the magnitude of the probable errors of observation apply with equal force to measures made under the microscope. Neither the probable error of a single observation nor the probable error of the mean of a given number of observations furnishes a safe criterion by which the real measure of accuracy may be estimated. For example, with the comparator for short lengths described in the April number of the *American Quarterly Microscopical Journal,* it is the experience of the writer that, in an unlimited number of repetitions of measures of the same space, the pointer will in every case fall upon the same tenth of a division of the index of the screw. Hence, if the readings are taken to tenths only, the resulting probable error will always be zero, without regard to the value of one division. In this particular instrument the value of one tenth of one division is one eighty-thousandth of an inch, but the probable error would still remain zero if the readings were carried to tenths of one division only for any change whatever in the pitch of the screw, and consequently for any reduction in the value of one division, provided the pointer always falls within this tenth.

The probable error of a *single* measure with the comparator for short lengths is about two millionths of an inch. If the probable error can be taken as a measure of precision, it ought not to be difficult to measure one millionth of an inch with entire certainty by repeating the measures a sufficient number of times.

Let us see if this theoretical accuracy is attainable. Before proceeding to the discussion, it may be worth while to say that a sharp distinction must be drawn between absolute accuracy and a superficial appearance of accuracy. If I determine the value of a centimeter within one ten-thousandth of its whole length, I can use the equivalent expression, one millionth of a meter; but it does not follow that I can measure a meter within this limit. I say that a given space, corresponding to one thousandth of an inch, requires a correction of one millionth of an inch; but it makes a wide difference whether I ascertain this fact by direct measurement, or whether I get it by dividing the correction for an entire inch by one thousand. Extending the number of figures in the quotient does not give a corresponding increase of accuracy. The index of the screw of my dividing engine can be set to correspond to a motion of one billionth of an inch with entire certainty as far as the mechanical indication of this degree of accuracy is concerned; yet previous to May, 1877, the actual errors of a given ruled plate amounted, under certain conditions, to as much as one seven-thousandth of an inch. Even now, after four epochs of improvement, I can hardly say of a given space that it is certainly true within one eighty-thousandth of an inch until a careful investigation has been made with the comparator. Again, it does not follow that, because the spaces of a closely ruled band of lines, like Nobert's bands, appear to be equal under an objective of high power, they are therefore to be taken as the measure of the real accuracy of the graduations. It is far more difficult to subdivide an inch into one hundred equal parts, than to make a further subdivision of one of these parts. As I shall presently show, almost all of the errors of a given graduation are periodic in their character, but the increments proceed by such minute variations in the case of closely ruled bands that they can only be detected when their sum amounts to an appreciable quantity Thus, if the accumulated error of a screw having a pitch of one in twenty amounts to one two-thousandth of an inch for half a revolution of the index, the average periodic error for each two-thousandth of an inch will be one hundred-thousandth of an inch. It will thus be seen that, for even the first of Nobert's bands, which are about ten thousand to the inch, the systematic error for any single space is inappreciable. But even in this case, only ten increments are required in order to produce an error of measurable magnitude.

A simple and direct way to determine the degree of precision with which measures under the microscope may be made, is to compare measurements of the same space made by different observers and under different conditions. I may get results which show an agreement *inter se* quite within the limits of the accuracy required, but which are yet wide of the truth. But if another equally skilful observer obtains substantially the same results from a series of measurements made under entirely different conditions, the inference of their general correctness may be drawn with tolerable safety.

In carrying forward this investigation I was fortunate in securing the co-operation of Professor Edward W. Morley, of Hudson, Ohio, whose paper will be found on page 164 of this volume of the Proceedings.

The rulings selected for joint measurement, are described as follows: —

Plate I. consists of eight bands. The first three bands are composed of twenty-six lines each. The distance between the lines is $\frac{1}{480}$ of an inch. The remaining five bands are composed of twenty-one lines each, the distance between the lines being $\frac{1}{600}$ of an inch. *All the rulings of this plate involve the periodic errors which belong to the ruling screw.*

Plate II. consists of three bands of very heavy lines, each band being composed of twenty-six lines. The interval between the lines is the same as in the corresponding three bands of Plate I. The lines are filled with graphite and are mounted in balsam. *In this plate the errors which are a function of one revolution of the screw were corrected during the process of ruling.*

Plate III. consists of 101 lines, separated by an interval of $\frac{1}{2400}$ of an inch, *and freed as nearly as possible from errors of all kinds.*

Plate IV. consists of 21 lines, separated by an interval of $\frac{1}{20}^{\text{mm.}}$, *corrected for systematic errors.*

The results given in the following tables under the head "Corr." represent the corrections which must be applied to each space of a given band in order to make it equal to a mean of all the spaces. They are *expressed in millionths of an inch*, except in Plate IV., in which the unit is *one hundred-thousandth of a millimeter*. The results given under the head Σ represent the accumulated errors reckoned from the first line of each band. In Plate I. the values given were formed by successive additions of the individual errors. In Plates II., III., and IV., the values in column Σ were obtained by measuring the accumulated errors directly with the comparator for short lengths, and the individual errors were found by successive subtractions.

The first band of Plate I. was measured with great care with a filar-micrometer made by Powell and Leland, with a glass eye-piece micrometer, with a comparator screw by Merz of Munich, and with the Clark screw mentioned above. The results from the Clark screw are somewhat discordant, as they were obtained before the instrument was fairly completed. They are, however, taken into account on the principle adopted of including *every* measure taken. The values of Plate III. were found by taking the mean of the accumulated errors of each successive group of five spaces, measured directly with the Clark comparator for short lengths. The separate results given under the first and fourth bands of Plate I. are given for the purpose of deducing the probable error of observation. They are not simple repetitions of measures made at one time. Each column refers to a different date. As the different sets of measures were only brought together from the note-books after all the work was done, I had no previous knowledge of the degree of agreement to be expected from separate measures of the same space. In fact, the comparison was made for the first time, soon after receiving the results communicated by Professor Morley.

PLATE I.—BAND I.

Space.	Corrections to each Space from Measures with Eye-piece Micrometer and ⅛-inch Objective.						Corrections to each Space from Measures with Merz Screw and ⅛-inch Objective.					Correction to each Space from Measures with Filar Micrometer and ⅛-inch Objective.		Correction to each Space from Measures with Clark Screw.	
	I.	II.	III.	IV.	Mean Corr.	Σ	I.	II.	III.	Mean Corr.	Σ	Corr.	Σ	Corr.	Σ
1	+3	—4	—6	+6	+0	+0	—8	—20	—6	—11	—11	+5	+5	—12	—12
2	—2	—4	—8	—14	—7	—7	—9	—6	—9	—8	—19	—11	—6	—13	—25
3	—14	—7	—11	—17	—12	—19	—17	—22	—11	—17	—36	—20	—26	—8	—33
4	—17	—18	—9	—17	—15	—34	—11	—21	—11	—14	—50	—13	—39	—23	—56
5	—19	—17	—14	—20	—18	—52	—25	—18	—16	—20	—72	—14	—53	—14	—70
6	—25	—25	—26	—23	—25	—77	—24	—16	—20	—20	—92	—28	—81	—28	—98
7	—25	—29	—21	—20	—24	—101	—28	—18	—16	—21	—113	—26	—107	—24	—122
8	—27	—27	—29	—26	—27	—128	—28	—18	—27	—24	—137	—24	—131	—11	—133
9	—22	—26	—25	—17	—25	—153	—21	—28	—30	—26	—163	—22	—153	—37	—170
10	—23	—14	—29	—17	—21	—174	—19	—14	—25	—19	—182	—18	—171	—8	—178
11	—21	—10	—21	—20	—18	—192	—5	—19	—12	—12	—194	—15	—186	—12	—190
12	—9	—8	—10	—11	—10	—202	+9	+0	—11	—1	—195	—7	—193	—11	—201
13	—12	—7	—8	—6	—8	—210	+23	—4	—3	+5	—190	+1	—192	+3	—198
14	+6	—1	+4	+6	+4	—206	+7	—4	+6	+3	—187	+6	—186	+8	—190
15	+9	+13	+8	+6	+9	—197	+0	+0	+9	+3	—184	+7	—179	+5	—185
16	+16	+18	+19	+12	+16	—181	+10	+16	+9	+12	—172	+12	—167	+17	—168
17	+21	+19	+22	+12	+19	—162	+16	+11	+11	+13	—159	+15	—152	+34	—134
18	+21	+21	+22	+18	+21	—141	+13	+26	+15	+18	—141	+16	—136	+7	—127
19	+25	+22	+22	+27	+24	—117	+20	+30	+29	+26	—115	+20	—116	+28	—99
20	+23	+23	+25	+24	+24	—93	+22	+30	+21	+24	—91	+25	—91	+20	—79
21	+23	+20	+24	+30	+24	—69	+23	+30	+28	+27	—64	+31	—60	+32	—47
22	+23	+23	+25	+27	+24	—45	+25	+25	+31	+27	—37	+21	—39	+12	—35
23	+16	+17	+19	+18	+18	—27	+12	+20	+18	+17	—20	+13	—26	+9	—26
24	+13	+17	+22	+18	+18	—9	+21	+15	+19	+18	—2	+21	—5	+26	—0
25	+7	+3	+3	+6	+5	—4	+0	+0	+0	+0	—2	+5	+0	+0	—0

PLATE I.

SPACE.	BAND II.		BAND III.		BAND IV.					FILAR MICROMETER.		BAND V.		BAND VI.		BAND VII.		BAND VIII.	
	Corr.	Σ	Corr.	Σ	I.	II.	III.	Mean Corr.	Σ	Corr.	Σ	Corr.	Σ	Corr.	Σ	Corr.	Σ	Corr.	Σ
1	+ 0	+ 0	+ 1	+ 1	+21	+14	+14	+16	+ 16	+19	+ 19	+ 7	+ 7	+ 4	+ 4	+ 4	+ 4	+ 3	+ 3
2	—11	— 11	— 6	— 5	— 2	— 7	—10	— 6	+ 10	— 7	+ 12	—12	— 5	—11	— 7	—13	— 9	—10	— 7
3	—18	— 29	—21	— 26	—17	—10	—22	—16	— 6	—14	— 2	—23	— 28	—17	— 24	—17	— 26	—18	— 25
4	—20	— 49	—19	— 45	—19	—25	—13	—19	— 25	—22	— 24	—33	— 61	—23	— 47	—27	— 53	—27	— 52
5	—24	— 73	—26	— 71	—28	—28	—33	—30	— 55	—34	— 58	—35	— 96	—32	— 79	—36	— 89	—37	— 89
6	—30	—103	—28	— 99	—32	—31	—33	—32	— 87	—34	— 92	—35	—131	—33	—112	—35	—124	—39	—128
7	—26	—129	—26	—125	—26	—27	—31	—28	—115	—25	—117	—31	—162	—29	—141	—32	—156	—30	—158
8	—24	—153	—29	—154	—29	—31	—33	—31	—146	—33	—150	—25	—187	—28	—169	—31	—187	—24	—182
9	—27	—180	—26	—180	—22	—23	—25	—23	—169	—24	—174	—18	—205	—19	—188	—23	—210	—23	—205
10	—22	—202	—25	—205	—11	—12	—10	—11	—180	—15	—189	—15	—220	—15	—203	—13	—223	— 9	—214
11	—16	—218	—17	—222	+ 0	— 7	+ 2	— 2	—182	+ 1	—188	+ 1	—219	+ 8	—195	— 2	—225	+ 2	—212
12	—11	—229	— 7	—229	+ 3	+ 6	+ 8	+ 6	—176	+ 4	—184	+16	—203	+10	—185	+15	—210	+12	—200
13	— 3	—232	— 1	—230	+15	+19	+20	+18	—158	+21	—163	+22	—181	+20	—165	+26	—184	+28	—172
14	+ 6	—226	— 1	—231	+20	+21	+20	+20	—138	+27	—136	+24	—157	+29	—136	+30	—154	+26	—146
15	+13	—213	+17	—214	+20	+28	+23	+24	—114	+26	—110	+27	—130	+28	—108	+29	—125	+34	—112
16	+23	—190	+20	—194	+27	+32	+32	+30	— 84	+33	— 77	+30	—100	+28	— 80	+30	— 95	+34	— 78
17	+27	—163	+25	—169	+29	+28	+29	+29	— 55	+29	— 48	+31	— 69	+27	— 53	+29	— 66	+27	— 51
18	+23	—140	+27	—142	+21	+29	+26	+25	— 30	+22	— 26	+26	— 43	+28	— 25	+27	— 39	+25	— 26
19	+19	—121	+25	—117	+21	+20	+23	+21	— 9	+21	— 5	+25	— 18	+19	— 6	+22	— 17	+21	— 5
20	+24	— 97	+27	— 90	+ 6	+ 6	+14	+ 9	+ 0	+ 5	+ 0	+17	— 1	+ 7	+ 1	+14	— 3	+ 8	+ 3
21	+22	— 75	+22	— 68															
22	+24	— 51	+24	— 44															
23	+23	— 28	+22	— 22															
24	+21	— 7	+19	— 3															
25	+ 6	— 1	+ 4	+ 1															

SPACE.	PLATE II.						PLATE III.		PLATE IV.	
	BAND I.		BAND II.		BAND III.		BAND I.		BAND I. $= \frac{1}{20}$ mm.	
	Corr.	Σ	Corr.	Σ	Corr.	Σ	Corr.	Σ	Corr.	Σ
1	+ 2	+ 2	— 6	— 6	— 3	— 3	+3	+3	+10	+10
2	— 4	— 2	— 1	— 7	+ 8	+ 5	+0	+3	—11	— 1
3	— 4	— 6	— 1	— 8	+ 3	+ 8	+0	+3	+ 0	— 1
4	+ 3	— 3	+ 4	— 4	—15	— 7	—4	—1	— 7	— 8
5	+ 3	+ 0	— 6	—10	+21	+14	+3	+2	+ 2	— 6
6	— 6	— 6	+ 1	— 9	+ 8	+22	—5	—3	+ 1	— 5
7	— 6	—12	+ 0	— 9	—22	+ 0	—1	—4	+ 3	— 2
8	—10	—22	— 8	—17	+ 9	+ 9	—2	—6	+ 2	— 0
9	+ 0	—22	— 2	—19	— 7	+ 2	—1	—7	+ 2	+ 2
10	+ 1	—21	— 8	—27	— 6	— 4	+0	—7	+ 5	+ 7
11	— 2	—23	— 5	—32	+ 9	+ 5	+0	—7	— 2	+ 5
12	— 8	—31	+ 3	—29	— 1	+ 4	+1	—6	+ 2	+ 7
13	+ 8	—23	+15	—14	— 6	— 2	+1	—5	+14	+21
14	+ 2	—21	— 5	—19	— 1	— 3	+1	—4	—17	+ 4
15	+ 0	—21	— 6	—25	+ 1	— 2	+5	+1	— 2	+ 2
16	+ 6	—15	+ 6	—19	+ 6	+ 4	+2	+3	— 1	+ 1
17	+ 9	— 6	— 2	—21	— 8	— 4	—6	—3	+ 0	+ 1
18	— 9	—15	+ 4	—17	+ 6	+ 2	+6	+3	+ 8	+ 8
19	+ 5	—10	+12	— 5	+ 1	+ 3	—1	+2	+ 0	+ 8
20	+10	+ 0	+ 4	— 1	+ 2	+ 5	+1	+3	—10	— 2
21	+ 0	+ 0	— 4	— 5	+12	+17				
22	+ 0	+ 0	— 1	— 6	— 5	+12				
23	+ 0	+ 0	— 4	—10	— 6	+ 6				
24	+ 2	+ 2	— 2	—12	— 8	— 2				
25	— 2	+ 0	+12	+ 0	+ 0	— 2				

From a comparison of the separate values obtained by myself and by Professor Morley, the following conclusions are drawn: —

(*a*) By comparing the separate values of Bands I. and IV. of Plate I., obtained with the eye-piece micrometer, with the corresponding mean values, the average probable error of the measure of a single space is found to be 19 ten-millionths of an inch, the greatest deviation from the mean in 156 measures being 8 millionths of an inch.

(*b*) Comparing with the mean value, the separate results obtained with the eye-piece micrometer, the Merz screw, and the filar micrometer, from the first and fourth bands of Plate I. we find the following average deviations : —

For the eye-piece micrometer,	17 ten-millionths of an inch.
For the Merz screw,	25 ten-millionths of an inch.
For the filar micrometer,	18 ten-millionths of an inch.

(*c*) Comparing the measures of the separate spaces made by Professor Morley with my own, made in the first band of Plate I. with the eye-piece micrometer, the Merz screw, and the filar-micrometer, and in the remaining bands of this plate with the eye-piece micrometer only, we find the following deviations expressed in millionths of an inch: —

Number of millionths,	0	1	2	3	4	5	6	7	8	9	10	11	12
Number of cases of agreement,	16	30	39	24	15	17	13	8	2	3	4	2	2

The mean deviation is 34 ten-millionths of an inch.

(*d*) Comparing the accumulated errors of the middle point obtained by Professor Morley and by myself, we have: —

	Band.	Rogers.	Morley.	R. — M.
Plate I.	1	—197	—174	—23
	2	—232	—218	—14
	3	—230	—204	—26
	4	—181	—177	— 4
	5	—219	—211	— 8
	6	—199	—199	+ 0
	7	—224	—233	+ 9
	8	—213	—237	+24
Plate II.	1	— 23	— 19	— 4
	2	— 14	— 53	+39
	3	— 2	— 23	+21
Plate III.	1	— 7	+ 31	—38

The numerical value of the average deviation is therefore 17 millionths of an inch. It is to be noted, however, that only the bands of Plate I. are strictly comparable, since these only were obtained in the same way, viz. by successive additions. The values for Plates II. and III., it will be remembered, were obtained in my own case by direct measurement, in which the degree of accuracy may be taken as nearly equal to that of the measure of any individual space, while those of Professor Morley were obtained by direct addition, in which case an error at any point is carried through the whole of the remaining series.

(*e*) In explanation of the disagreement in the maximum values of the accumulated errors, even in the bands of Plate I., which were ruled at the same time, it is to be said that the graduation was done before I had learned the necessity of dispensing with oil or grease as a lubricant. Without doubt, a part of the discordance is due to errors of measurement, but recent experience has convinced me that, when oil is used as a lubricant, every precision-screw has a variable periodic error, depending on the position of the nut in the line of its motion, especially when there is a decided change of temperature. It is sufficient to say here, that, since adopting a substitute for oil, the errors of the screw have remained practically constant.

(*f*) It will be seen from simple inspection that nearly all of the errors under discussion are periodic in their character. This may be shown conclusively in the following way: —

If we have a series of errors depending on one revolution of the screw, which are *strictly periodic in their character*, and from which all accidental errors are excluded, the given series can be represented *exactly* by a series of equations of the form, —

$$n = + a \sin x + b \cos x + a' \sin 2x + b' \cos 2x, \text{ \&c.};$$

in which

n represents any given value of the series;
x is an aliquot part of one revolution of the screw;
a, b, a', b', are unknown coefficients.

If, therefore, an expression of this form is found which will represent all the given errors of a series, it may be safely affirmed that the errors themselves are entirely periodic in their character.

Making n equal, successively, to the mean of the values of the individual errors of spaces 1, 2, 3, &c. of the first three bands of Plate I., we have a series of equations whose solution by least squares will give the normal equation,

$$n = -26.6 \sin x + 5.9 \cos x - 0.1 \sin 2x + 0.8 \cos 2x.$$

In like manner we get from the mean of the separate values of Bands IV., V., VI., VII., VIII. of the same plate,

$$n = -29.1 \sin x + 9.9 \cos x - 2.0 \sin 2x + 1.3 \cos 2x.$$

Substituting in these equations the known values of x, we get values of n which are directly comparable with the observed values given in the tables. The observed and the computed values are given in the following table, as well as the deviations of both the individual and the accumulated errors from the computed values.

Plate I. — Bands I., II., III.					
Individual Errors.			Accumulated Errors.		
Observed value of *n*.	Computed value of *n*.	Δ	Observed.	Computed.	Δ
— 1	+ 0	—1	— 1	+ 0	—1
— 9	— 8	—1	— 10	— 8	—2
—18	—14	—4	— 28	— 22	—6
—19	—19	+0	— 47	— 41	—6
—23	—24	+1	— 70	— 65	—5
—27	—28	+1	— 97	— 93	—4
—25	—28	+3	—122	—121	—1
—25	—27	+2	—147	—148	+1
—27	—25	—2	—174	—173	—1
—21	—21	+0	—195	—194	—1
—16	—15	—1	—211	—209	—2
— 8	— 8	+0	—219	—217	—2
— 1	— 2	+1	—220	—219	—1
+ 4	+ 6	—2	—216	—213	—3
+12	+12	+0	—204	—201	—3
+19	+16	+3	—185	—185	+0
+24	+22	+2	—161	—163	+2
+22	+24	—2	—139	—139	+0
+23	+26	—3	—116	—113	—3
+25	+26	—1	— 91	— 87	—4
+24	+26	—2	— 67	— 61	—6
+23	+22	+1	— 44	— 39	—5
+20	+18	+2	— 24	— 21	—3
+20	+14	+6	— 4	— 7	+3
+ 4	+ 7	—3	+0	+ 0	+0

Plate I.— Bands IV., V., VI., VII., VIII.					
Individual Errors.			Accumulated Errors.		
Observed value of *n*.	Computed value of *n*.	Δ	Observed.	Computed.	Δ
+ 7	+ 1	+6	+ 7	+ 1	+6
—11	— 9	—2	— 4	— 8	+4
—16	—18	+2	— 20	— 26	+6
—26	—26	+0	— 47	— 52	+5
—34	—31	—3	— 81	— 83	+2
—35	—33	—2	—116	—116	+0
—30	—32	+2	—147	—148	+1
—27	—28	+1	—174	—176	+2
—21	—21	+0	—195	—196	+1
—13	—11	—2	—208	—207	—1
+ 2	+ 0	+2	—207	—207	+0
+12	+ 9	+3	—195	—198	+3
+23	+18	+5	—172	—180	+8
+26	+27	—1	—146	—153	+7
+29	+31	—2	—118	—122	+4
+31	+33	—2	— 87	— 89	+2
+29	+32	—3	— 58	— 57	—1
+26	+27	—1	— 33	— 30	—3
+22	+20	+2	— 11	— 10	—1
+11	+11	+0	+ 0	+ 1	—1

Finally, we have a severe test in the agreement of the values of n computed for parts of the revolution which were not observed. In order to adapt the equations, for example, to successive ten degrees of revolution, the coefficients of the first equation must be multiplied by $\frac{10}{14.4} = .694$, and those of the second equation must be multiplied by $\frac{10}{18} = .556$. We shall then have: —

$$(1)\quad n = -18.5 \sin x + 4.1 \cos x - 0.1 \sin 2x + 0.6 \cos 2x;$$

$$(2)\quad n = -16.2 \sin x + 5.4 \cos x - 1.1 \sin 2x + 0.7 \cos 2x.$$

Substituting $x = 0°, 10°, 20°$, &c. in these equations, and comparing the results, we have the following discordances expressed in millionths of an inch: —

Number of millionths,	0	1	2	3	4
Number of cases,	5	7	17	3	4

The average deviation is 19 ten-millionths of an inch, and there are only 7 cases in which the disagreement exceeds 2 millionths of an inch.

(*g*) The relative advantages of the eye-piece micrometer, the filar micrometer, and the screw comparator, for narrow intervals, are nearly equal, as will be seen from the following comparison of the individual values derived by each method of observation, with the normal values found from the equation

$$n = -23.8 \sin x + 5.4 \cos x - 0.1 \sin 2x + 1.0 \cos x,$$

which represents the mean curve for the first band of Plate I.

Number of millionths,		0	1	2	3	4	5	6	7	8	9	10
Number of cases,	Eye-piece micrometer,	7	8	3	5	0	1	0	1	0	0	0
	Filar-micrometer,	4	2	3	5	2	1	5	1	2	0	0
	Merz screw,	4	2	4	2	5	2	2	2	0	1	1

(*h*) It appears from this investigation that it is possible to reduce the errors of a precision-screw for short intervals to about one hundred-thousandth of an inch by applying the corrections derived from the equation which represents the periodic errors. Since the rejection of oil as a lubricant, the errors have been considerably reduced.

(*i*) In a meridian circle having a diameter of 30 inches, one second of arc is equal to .0000727 of an inch. It appears, therefore, from this investigation, that, even if the attached microscopes have the same power as those used in this investigation, the ultimate limit of accuracy in the matter of bisection and reading only, must be at least 0.″05. But the microscopes of the meridian circle of Harvard College Observatory magnify only 51 diameters, while the magnifying powers used in this series of measures were 194, 290, 560, and 870. Moreover, this limit has reference only to repeated readings of the microscopes for the same position of the instrument. It has, therefore, only a *relative* value. When, in addition to the errors of simple pointing and reading, we take into account the accidental and the systematic errors of division in the graduated circles and the outstanding errors always found in measures of large arcs of a circle, the present limit of precision cannot fall much below 0.″2.

Since the completion of this investigation a further opportunity of comparing the results of measures of the same intervals by different observers has occurred. Through the kindness of Professor George F. Barker, of the University of Pennsylvania, I obtained the loan of a ruled plate from the precision-screw of Mr. L. M. Rutherfurd. This plate is marked "$\frac{68}{360}$ rev." It consists of 11 lines, covering a space as nearly equal to one millimeter as the even notches of the index of the screw will give this value. A transverse line subdivides the vertical lines. The lines are apparently filled with graphite, and they are protected by a coating of transparent varnish.

I first measured the ten spaces of this plate in May, 1878. The plate was then sent to Professor Morley. After its return, and before comparing the results already obtained, it was again measured. In April and May, 1879, still another series of measures was made. The plate was then placed in the hands of Mr. J. R. Edmands, who is a skilful and careful observer with the microscope.

The measures made by Professor Morley and myself were of the separate spaces, and the accumulated errors were found by successive additions. Mr. Edmands made some measures in this way, but he also measured the distance of the successive lines from each of the end lines. The differences between the errors of the adjacent lines were then compared with the direct measures of the spaces, while the sums of the errors of the spaces were compared with the errors of the individual lines. The values given by him were obtained by giving to each determination its proper relative weight.

In accordance with the prearranged plan of observation, each observer remained in ignorance of the results obtained by the other observers until the work of measurement was completed. The following are the values of the corrections required to reduce each space of this particular millimeter to the mean of all the spaces. They are expressed in *millionths of a centimeter.*

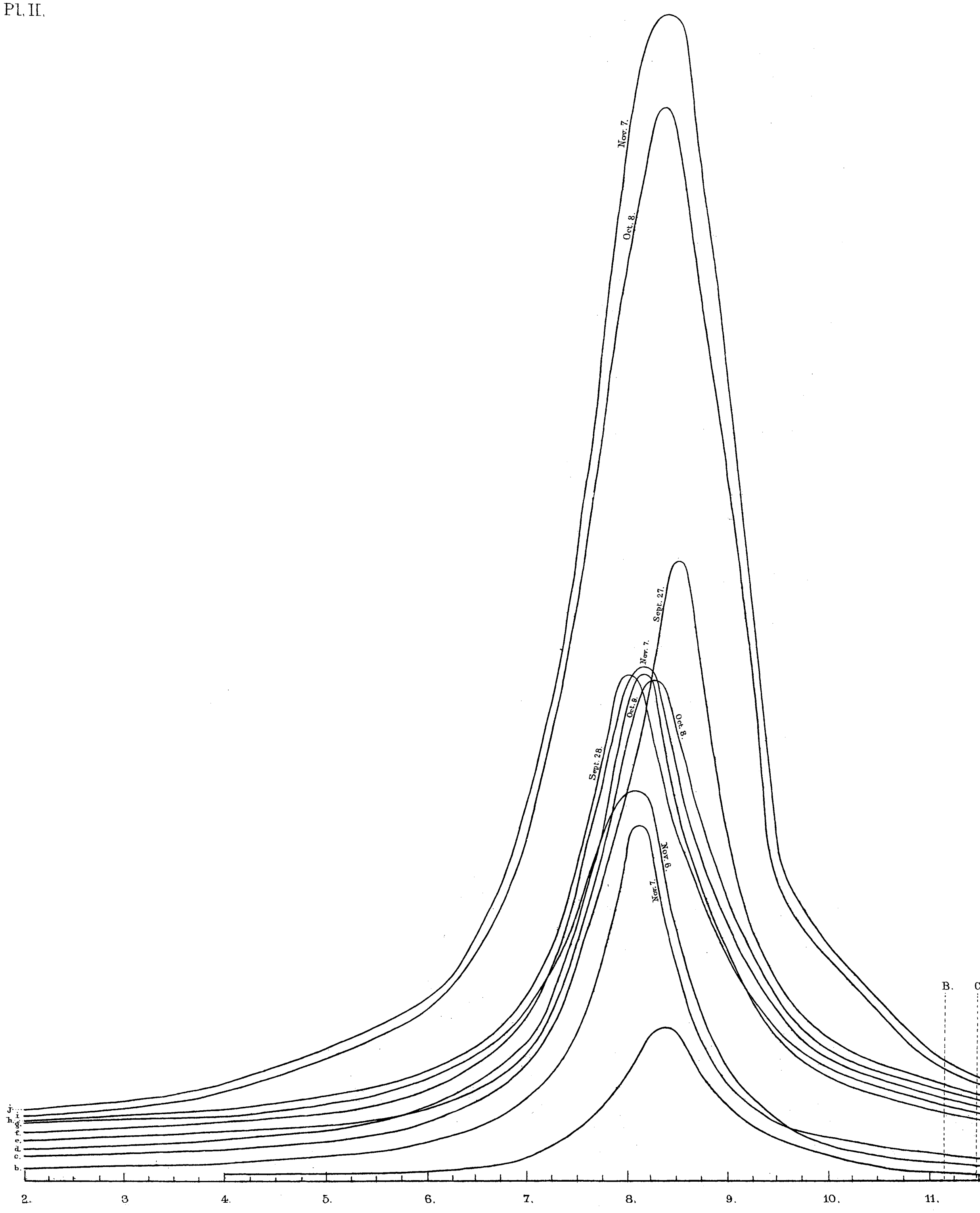
INDIVIDUAL ERRORS.

SPACES.	ROGERS. May, 1878.	MORLEY.	ROGERS. October and November, 1878.	ROGERS. April and May, 1879.	EDMANDS.	Mean Values.
1	—36	— 8	—37	—16	—14	—22
2	—14	—12	—11	—19	—16	—14
3	— 6	—10	—18	—27	—16	—15
4	— 4	— 1	— 2	+11	+ 2	+ 1
5	— 7	— 1	— 3	— 1	— 3	— 3
6	—28	— 4	—23	— 4	— 3	—12
7	+ 0	— 1	+ 2	— 1	— 1	+ 0
8	+ 8	— 3	+ 4	+ 2	— 3	+ 2
9	+35	+ 3	+33	+13	+10	+19
10	+52	+35	+55	+42	+44	+46
ACCUMULATED ERRORS.						
1	—36	— 8	—37	—16	—14	—22
2	—50	—20	—48	—35	—30	—37
3	—56	—30	—66	—62	—46	—51
4	—60	—31	—68	—51	—44	—51
5	—67	—32	—71	—52	—47	—54
6	—95	—36	—94	—56	—50	—66
7	—95	—37	—92	—57	—51	—66
8	—87	—40	—88	—55	—54	—65
9	—52	—37	—55	—42	—44	—46
10	— 0	— 2	— 0	— 0	— 0	— 0

It is somewhat doubtful whether the mean values given, represent the actual errors of the spaces. There are some indications that a shrinkage of the film of varnish has occurred since it was first applied, and this action may have produced some effect upon the graphite with which the lines are filled. Although graphite is an impalpable powder, I have seen many instances in which it has been lifted in mass from the filled lines and thrown a distance as great as one thousandth of an inch without breaking the continuity of the particles. This action seems to take place only when the lines and the filling are protected by a thin cover-glass, closely cemented to the slip on which the lines are ruled. Sometimes an explosion seems to take place, scattering the graphite in all directions, leaving it in curves having nearly a uniform shape. I have never been fortunate in seeing this action, but in the case of one ruled plate an actual observation limits the time within which the explosion must have occurred to about ten days. In this case the lines remained perfect for about four months after they were ruled and filled, but between the 1st and the 10th of April of the present year nearly one half of the powder was completely removed from the lines.

It is possible also, that some of the discordances are due to the fact that the measures were not all made along the same horizontal line. My earlier measures were made along a line just above the transverse line, while the later ones were made along a line a little below the transverse line. My observations of April and May of the present year, and those made by Mr. Edmands, ought to be comparable, since they were made at nearly the same time and along the same line. The greatest disagreement is in the third space; and that the discordance is not an accidental one is shown by the fact that *all* my observations agree in giving —27, while *all* of his agree in giving —16. But, admitting that the discordances are all due to errors of observation, it will be seen that the average deviation from the mean is only 6 millionths of a centimeter.

Pl. II.



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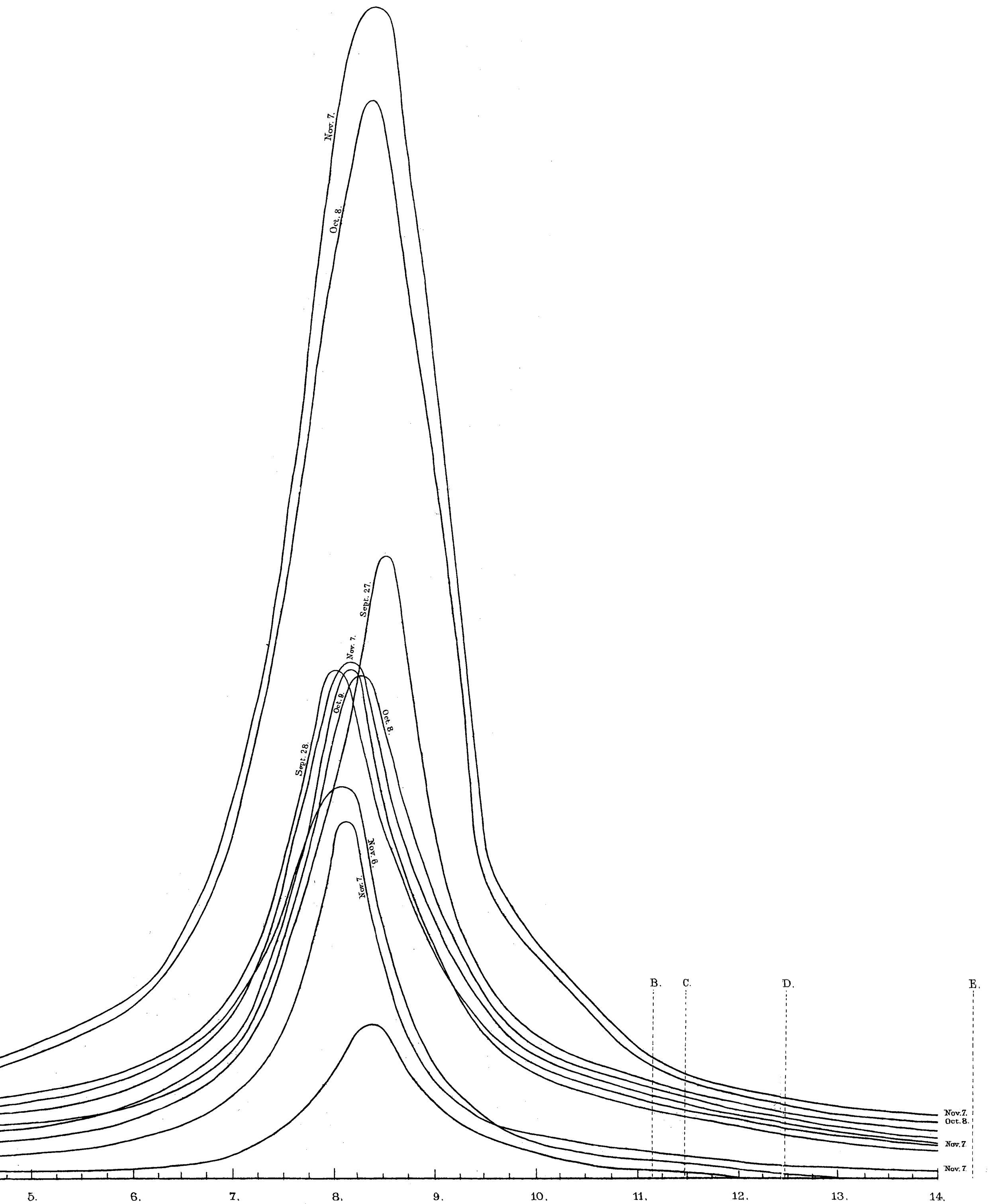
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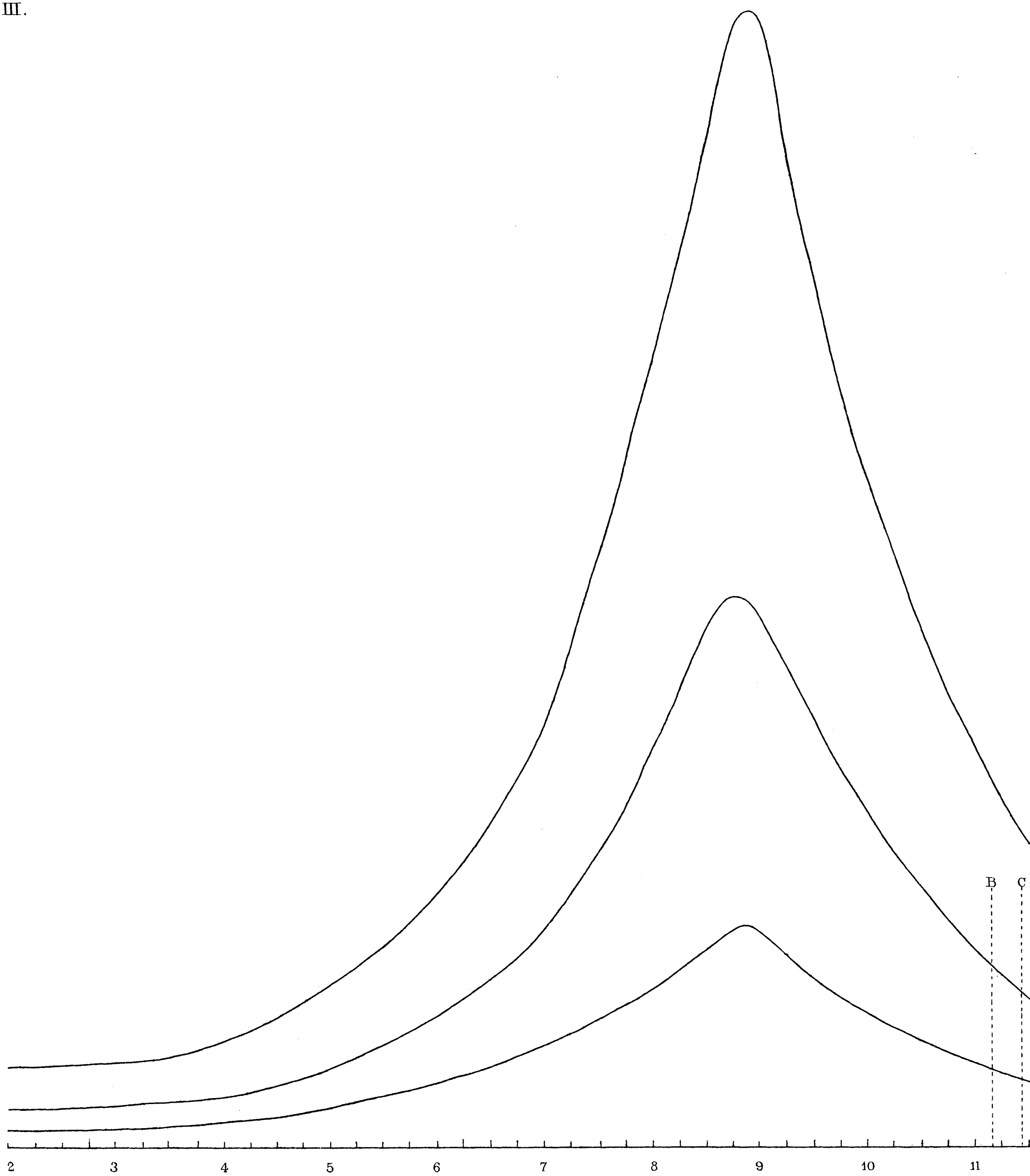
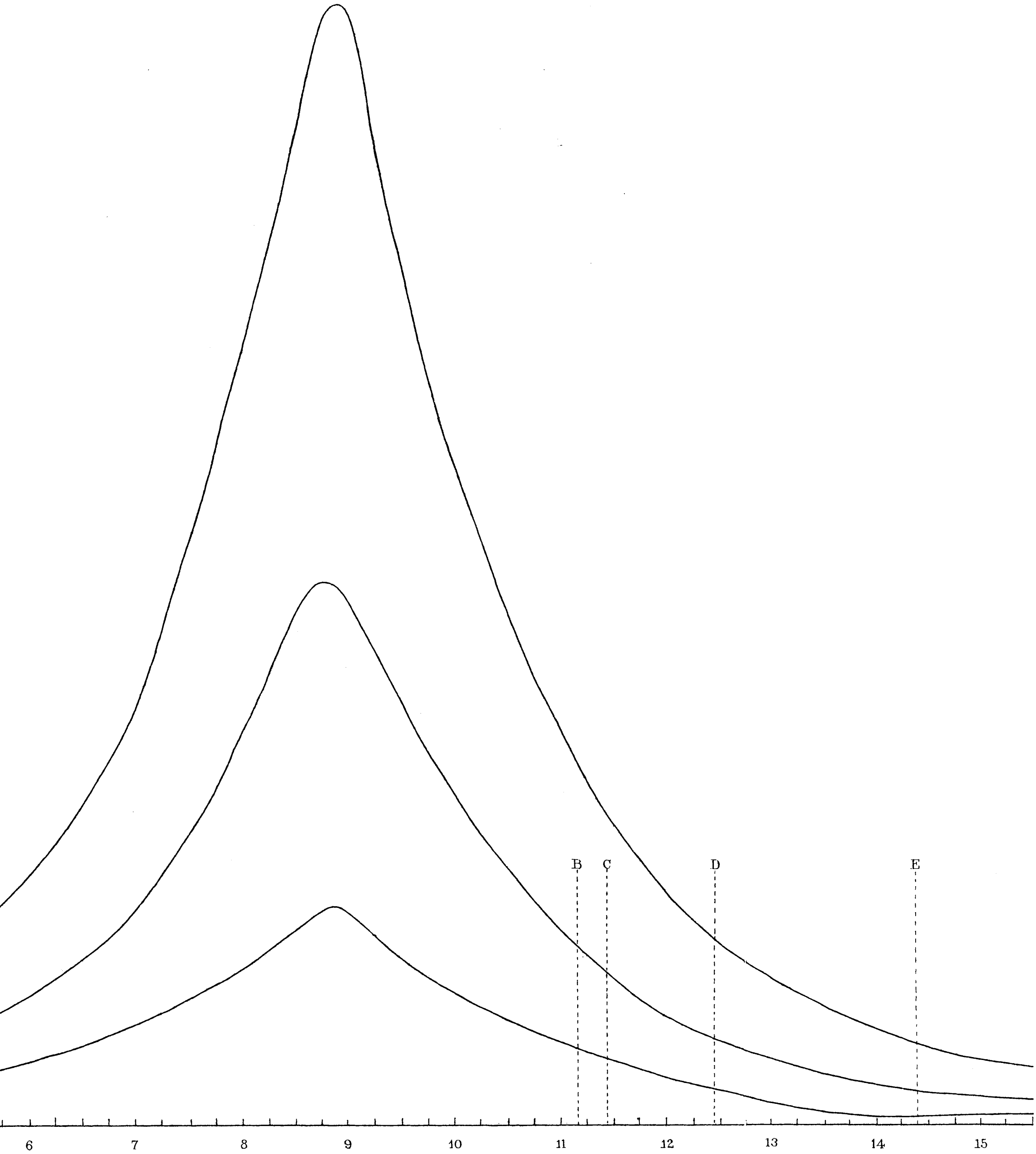
Pl. III.

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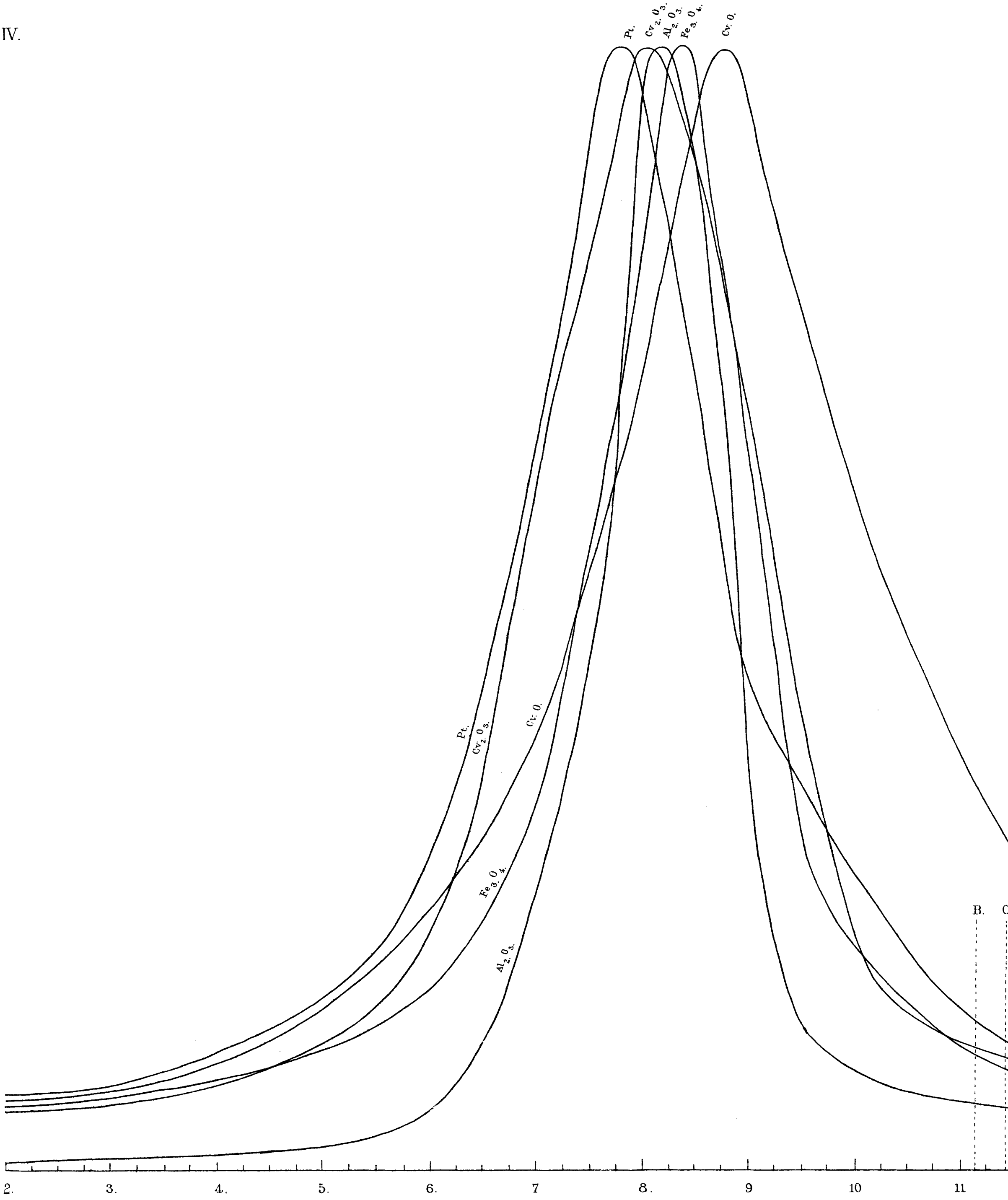
Pl.IV.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$Al_2.O_3.$
$Fe_3.O_4.$
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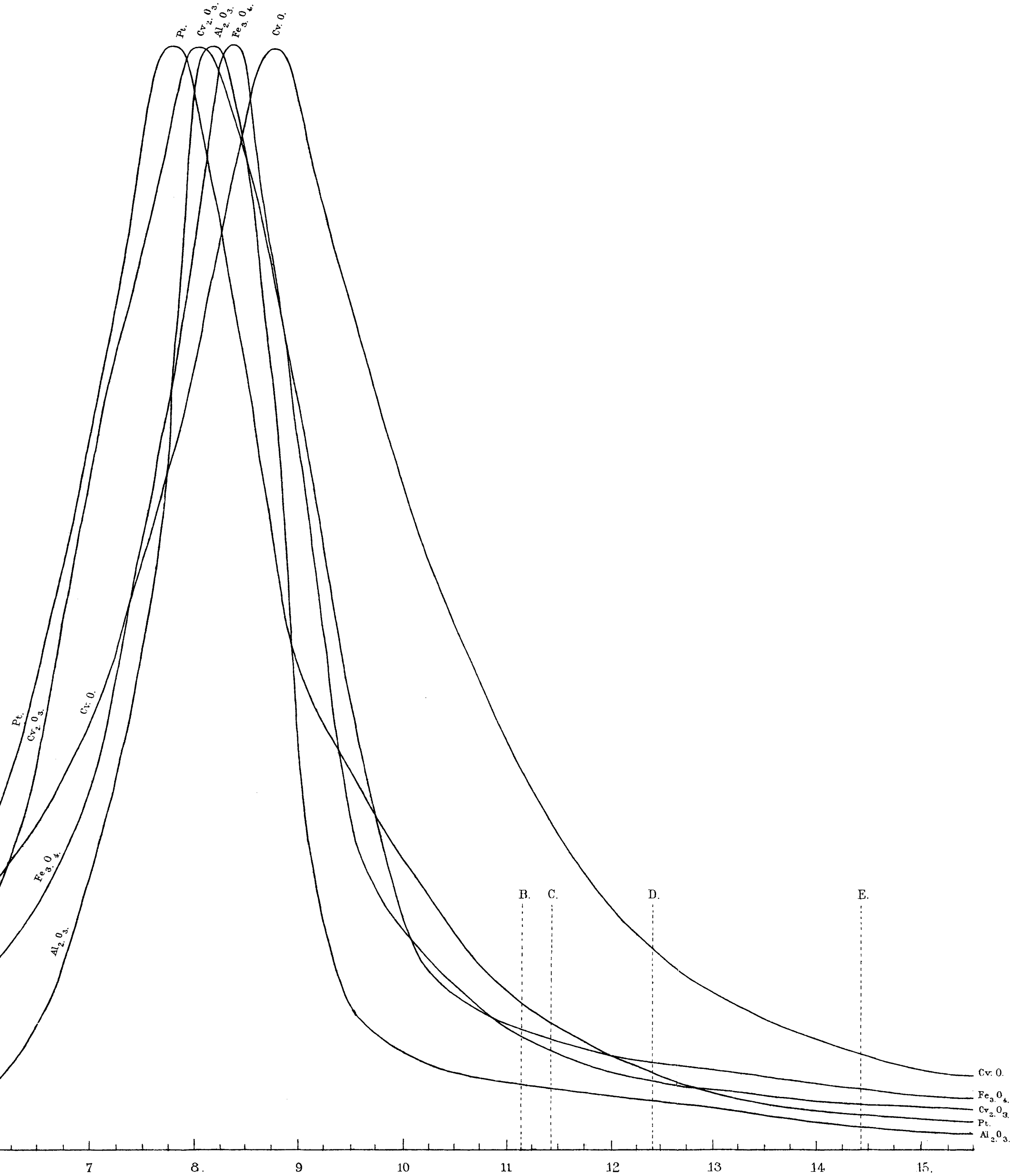
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